Substituting (5) and (6) for W and V_2 in (4), we have

$$\frac{Q}{R} = A\frac{\iiint_V E \cdot E^* \, dv}{\left(\int_a^b E \cdot dl\right)^2} \tag{7}$$

where $A = \omega\varepsilon$, a constant at a given ω and ε. (7) shows that the only requirement to guarantee the proportionality of Q and R, or the validity of (1) is

$$\frac{\iiint_V E \cdot E^* \, dv}{\left(\int_a^b E \cdot dl\right)^2} = B = \text{constant.} \tag{8}$$

In the following, we are to prove that (8) holds no matter how large the Q variation so long as such a change is caused only by a change in the power loss, P_L.

According to [3], P_L and W are both proportional to the square of the field strength; thus P_L at any instant of time is proportional to W, i.e.,

$$P_L = -\frac{dW}{dt} = 2\alpha W \tag{9}$$

where α is an attenuation factor.

The solution of (9) is

$$W = W_0 e^{-2\alpha t} \tag{10}$$

and

$$Q = \frac{\omega W}{P_L} = \frac{\omega}{2\alpha}. \tag{11}$$

Equation (11) shows that the Q is determined by α for a given ω. Because of (10), E at this instant can also be expressed as

$$E = E_0 e^{-\alpha t} \tag{12}$$

where E_0 is the spatial distribution of E. Substituting (12) for E in (8), we obtain

$$B = \frac{\iiint_V E \cdot E^* \, dv}{\left(\int_a^b E \cdot dl\right)^2} = \frac{e^{-2\alpha t}\iiint_V E_0 \cdot E_0^* \, dv}{e^{-2\alpha t}\left(\int_a^b E_0 \cdot dl\right)^2} = B_0 \tag{13}$$

which indicates that the constant B is not a function of α, and, therefore, not a function of Q either for a given frequency. Consequently, any Q variation caused by a power loss variation in the cavity would not alter the constant C in (1). Therefore, (1) holds over an unlimited Q range.

In practice, Q variation can be caused by the variation in the power loss and by the variation in the field spatial distribution, E_0. For the former case, we have proved (1) to hold, However, for the latter case, (1) may or may not be valid depending on whether the constant B is altered as a result of the variation in E_0, for, in essence, validation of (1) only requires invariance of the constant B. Note that B is a ratio of two integrals of E_0; a change in E_0, particularly, a small local change as in all perturbations, does not necessarily change this ratio. Therefore, a linear relation between Q and R holds not only in the situations where the Q variation is caused by the variation in power loss, but also in many situations where the field distribution has somehow been changed but the constant B does not. We have recently applied this concept in the design of a self-heating single-frequency high temperature dielectrometer with excellent results.

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Transient Analysis of Lossy Coupled Transmission Lines in a Lossy Medium Using the Waveform Relaxation Method

F. C. M. Lau and E. M. Deeley

Abstract—**The waveform relaxation method has been shown to be both efficient and accurate when applied to coupled transmission lines with conductor losses. In this paper, the method is generalized to include the dielectric loss surrounding the transmission lines. The distributed loss model assumes that the conductance matrix is approximately diagonal and its product with the resistive matrix is a scalar matrix. Computational results using the model is presented and compared with HSPICE solutions.**

I. Introduction

Recently, the method of characteristics has been generalized by Chang [1] for waveform relaxation analysis so that time-domain simulations of lumped-parameter networks interconnected with coupled transmission lines can be carried out more efficiently. It has been shown by the present authors [2] that solution problems related to the presence of dc components can arise, leading to a complete breakdown of the iterative process, and a modified iterative algorithm has been proposed to overcome these problems. In this paper, the dielectric leakage of the medium in which the transmission lines are embedded is taken into consideration.

II. Coupled Lines with Particular Conductor and Dielectric Loss Matrices

Under general conditions, the voltages and currents along a set of lossy coupled transmission lines, each of length l, are described by

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the generalized telegraphist's equation:

$$\frac{\delta \boldsymbol{v}(x,t)}{\delta x} = -\boldsymbol{L}\frac{\delta \boldsymbol{i}(x,t)}{\delta t} - \boldsymbol{R}\boldsymbol{i}(x,t) \quad (1a)$$

$$\frac{\delta \boldsymbol{i}(x,t)}{\delta x} = -\boldsymbol{C}\frac{\delta \boldsymbol{v}(x,t)}{\delta t} - \boldsymbol{G}\boldsymbol{v}(x,t) \quad (1b)$$

where $\boldsymbol{v}(x,t)$ and $\boldsymbol{i}(x,t)$ are column vectors defining the voltages $v_k(x,t)$ and currents $i_k(x,t)$ on the conductors $k = 1, 2, \ldots, n$. $\boldsymbol{R}$ is the diagonal matrix of the per-unit-length (PUL) resistance of the conductors. $\boldsymbol{G}$ is the nxn symmetric matrix of the PUL conductance of the surrounding dielectric medium. $\boldsymbol{L}$ and $\boldsymbol{C}$ are the nxn symmetric matrices of the PUL inductance and capacitance, and:

$$\boldsymbol{LC} = (1/v^2)\boldsymbol{I}_n \quad (2)$$

where $\boldsymbol{v}$ is the wave propagation velocity and $\boldsymbol{I}_n$ is the nth-order identity matrix, assuming the medium is homogeneous.

A. Analysis in the S-Domain

If the dielectric loss to ground is much more significant than the loss between the lines, the dielectric matrix $\boldsymbol{G}$ in (1b) will be nearly diagonal. If, furthermore, the product of the matrices $\boldsymbol{R}$ & $\boldsymbol{G}$ is a scalar matrix (multiple of the identity matrix), which is always the case for identical lines, i.e.

$$\boldsymbol{RG} = m\boldsymbol{I}_n \quad (3)$$

it can be proved that the lossy stripline system is equivalent to a set of decoupled *RLCG* transmission lines connected with congruence transformers. One system satisfying such conditions is a set of lossy coupled transmission lines running parallel to each other. The configurations of the lines are identical and the lines are the same distance from the ground plane. Under such conditions, the conductor loss and the dielectric loss to ground are the same for each line. If the dielectric loss between the lines is negligible, the relation $\boldsymbol{RG} = m\boldsymbol{I}_n$ will also be satisfied.

Using the decoupling procedures in [1] to evaluate the transformation matrix $[\boldsymbol{X}]$ and apply it to (1) with (2) and (3), the following equations are obtained:

$$\frac{\delta \boldsymbol{e}(x,t)}{\delta x} = -\tilde{\boldsymbol{L}}\frac{\delta \boldsymbol{j}(x,t)}{\delta t} - \tilde{\boldsymbol{R}}\boldsymbol{j}(x,t) \quad (4a)$$

$$\frac{\delta \boldsymbol{j}(x,t)}{\delta x} = -\tilde{\boldsymbol{C}}\frac{\delta \boldsymbol{e}(x,t)}{\delta t} - \tilde{\boldsymbol{G}}\boldsymbol{e}(x,t) \qquad (0 \le x \le l) \quad (4b)$$

which describe a set of n decoupled $RLCG$ transmission lines and

$$\tilde{\boldsymbol{R}} = \mathrm{diag}(L_k|\tau_k) \qquad \tilde{\boldsymbol{L}} = \mathrm{diag}(L_k)$$
$$\tilde{\boldsymbol{C}} = \mathrm{diag}\left(\frac{1}{v^2 L_k}\right) \qquad \tilde{\boldsymbol{G}} = \mathrm{diag}\left(\frac{m\tau_k}{L_k}\right) \quad (5)$$

are all diagonal matrices, where $\{\tau_k\}$ are the eigenvalues of the matrix $\boldsymbol{R}^{-1/2}\boldsymbol{L}\boldsymbol{R}^{-1/2}$, and $\boldsymbol{R}^{-1/2} = \mathrm{diag}(R_k^{-1/2})$.

Assuming the lines are terminated in Thevenin equivalent circuits, it can be shown that the transmission-line terminal voltages are given by

$$\boldsymbol{V}_A = (\boldsymbol{I}_n - \boldsymbol{P})^{-1}\boldsymbol{U}_A \quad (6a)$$
$$= \boldsymbol{U}_A + \boldsymbol{P}\boldsymbol{U}_A + \boldsymbol{P}^2\boldsymbol{U}_A + \boldsymbol{P}^3\boldsymbol{U}_A + \cdots \quad (6b)$$
$$\boldsymbol{V}_D = (\boldsymbol{I}_n - \boldsymbol{Q})^{-1}\boldsymbol{U}_D \quad (7a)$$
$$= \boldsymbol{U}_D + \boldsymbol{Q}\boldsymbol{U}_D + \boldsymbol{Q}^2\boldsymbol{U}_D + \boldsymbol{Q}^3\boldsymbol{U}_D + \cdots \quad (7b)$$

where

$$\boldsymbol{P} = (\boldsymbol{I}_n + \boldsymbol{\rho}_A)\boldsymbol{\Phi}\boldsymbol{\rho}_D\boldsymbol{\Phi}\boldsymbol{\rho}_A(\boldsymbol{I}_n + \boldsymbol{\rho}_A)^{-1} \quad (8a)$$
$$\boldsymbol{Q} = (\boldsymbol{I}_n + \boldsymbol{\rho}_D)\boldsymbol{\Phi}\boldsymbol{\rho}_A\boldsymbol{\Phi}\boldsymbol{\rho}_D(\boldsymbol{I}_n + \boldsymbol{\rho}_D)^{-1} \quad (8b)$$
$$\boldsymbol{U}_A = (1/2)[(\boldsymbol{I}_n + \boldsymbol{P}\boldsymbol{\rho}_A^{-1})(\boldsymbol{I}_n - \boldsymbol{\rho}_A)\boldsymbol{E}_X + (\boldsymbol{I}_n + \boldsymbol{\rho}_A)\boldsymbol{\Phi}(\boldsymbol{I}_n - \boldsymbol{\rho}_D)\boldsymbol{E}_Y] \quad (8c)$$
$$\boldsymbol{U}_D = (1/2)[(\boldsymbol{I}_n + \boldsymbol{\rho}_D)\boldsymbol{\Phi}(\boldsymbol{I}_n - \boldsymbol{\rho}_A)\boldsymbol{E}_X + (\boldsymbol{I}_n + \boldsymbol{Q}\boldsymbol{\rho}_D^{-1})(\boldsymbol{I}_n - \boldsymbol{\rho}_D)\boldsymbol{E}_Y] \quad (8d)$$
$$\boldsymbol{\rho}_A = (\boldsymbol{Z}_X - \boldsymbol{Z}_0)(\boldsymbol{Z}_X + \boldsymbol{Z}_0)^{-1} \quad (8e)$$
$$\boldsymbol{\rho}_D = (\boldsymbol{Z}_Y - \boldsymbol{Z}_0)(\boldsymbol{Z}_Y + \boldsymbol{Z}_0)^{-1}. \quad (8f)$$

The characteristic impedance matrix and the exponential propagation matrix of the lossy stripline system are respectively

$$\boldsymbol{Z}_0 = \boldsymbol{X}\mathrm{diag}(Z_{0k})\boldsymbol{X}^t \quad (9a)$$
$$\boldsymbol{\phi} = \boldsymbol{X}\,\mathrm{diag}[\exp(-\theta_k)]\boldsymbol{X}^{-1} \quad (9b)$$

where the characteristic impedances $\{Z_{0k}\}$ and the propagation functions $\{\theta_k\}$ are defined in terms of the decoupled transmission-line parameters:

$$Z_{0k} = \sqrt{(R_k + sL_k)/(G_k + sC_k)} \quad (9c)$$
$$\theta_k = \sqrt{(R_k + sL_k)(G_k + sC_k)}l. \quad (9d)$$

B. Waveform Relaxation Algorithm

The waveform relaxation algorithm used here is similar to that described in [1] and [2] with some differences. The decoupled transmission lines in the equivalent circuit are now of *RLCG* type instead of *RCL* type. As a result, the characteristic impedances $\{Z_{0k}\}$ and the propagation functions $\{\theta_k\}$ have different expressions, as given in (9) . Each of the decoupled lines is transformed into the equivalent model as shown in Fig. 2 and each of the characteristic impedances $\{Z_{0k}\}$ is modelled by a Pade approximation as described in [3]. The waveform relaxation algorithm used to simulate such a system is now explained.

Step 1: Given a set of coupled transmission lines with conductor and dielectric loss matrices satisfying $\boldsymbol{RG} = m\boldsymbol{I}_n$, and terminated in Thevenin equivalent circuits $\{\boldsymbol{E}_X, \boldsymbol{Z}_X\}$ & $\{\boldsymbol{E}_Y, \boldsymbol{Z}_Y\}$, an initial dc analysis [3] is first performed (since $\boldsymbol{R}$ is diagonal, so is $\boldsymbol{G}$) to derive the initial terminal voltages and currents $\boldsymbol{v}_{a0}$, $\boldsymbol{v}_{d0}$, $\boldsymbol{i}_{a0}$, $\boldsymbol{i}_{d0}$, where the subscripts a and d refer to the near-end and far-end terminals of the stripline system respectively.

Step 2: Decouple the stripline system by placing congruence transformers at the ends. Replace each of the n decoupled *RLCG* transmission lines by an equivalent disjoint 2-port network. The equivalent system is shown in Fig. 2, where k is the iteration counter. The lossy characteristic impedances are synthesized by ladder networks [3] with circuit elements defined in terms of the decoupled lossy transmission-line parameters.

Step 3: The two-part network reproduces the exact initial terminal characteristics by assigning the FFT waveform generators with the initial values

$$\boldsymbol{w}_{a0} = \boldsymbol{X}^{-1}\boldsymbol{v}_{a0} - \mathrm{diag}(H_\imath)\boldsymbol{X}^t\boldsymbol{i}_{a0}$$
$$\boldsymbol{w}_{d0} = \boldsymbol{X}^{-1}\boldsymbol{v}_{d0} + \mathrm{diag}(H_\imath)\boldsymbol{X}^t\boldsymbol{i}_{d0}.$$

where $H_\imath$ is the dc resistance of the characteristic impedance of the i th decoupled tranmission line.

Step 4: Initialize the iteration counter ($k = 1$) and the FFT waveform generators:

$$w_{a\imath}^{(0)}(t) = w_{\imath a0}u(t) \qquad i = 1, 2, \cdots, n$$

where $w_{\imath a0}$ is the $\imath$ th element of the vector w_{a0} and $u(t)$ is the step function.

Step 5: Connect the Thevenin equivalent circuit $\{\boldsymbol{E}_X, \boldsymbol{Z}_X\}$ to Subcircuit I and carry out the transient analysis for the entire time interval ($0 \le t \le T$) to obtain the terminal voltage waveforms $\{\boldsymbol{v}_a^{(k)}(t)\}$.

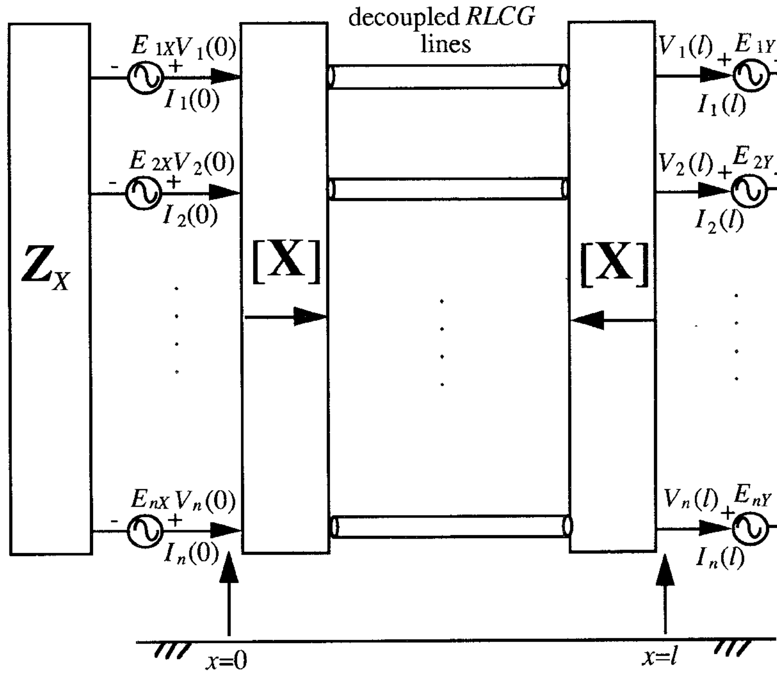

Fig. 1. The decoupled equivalent circuit of an n-conductor lossy coupled transmission-line system.

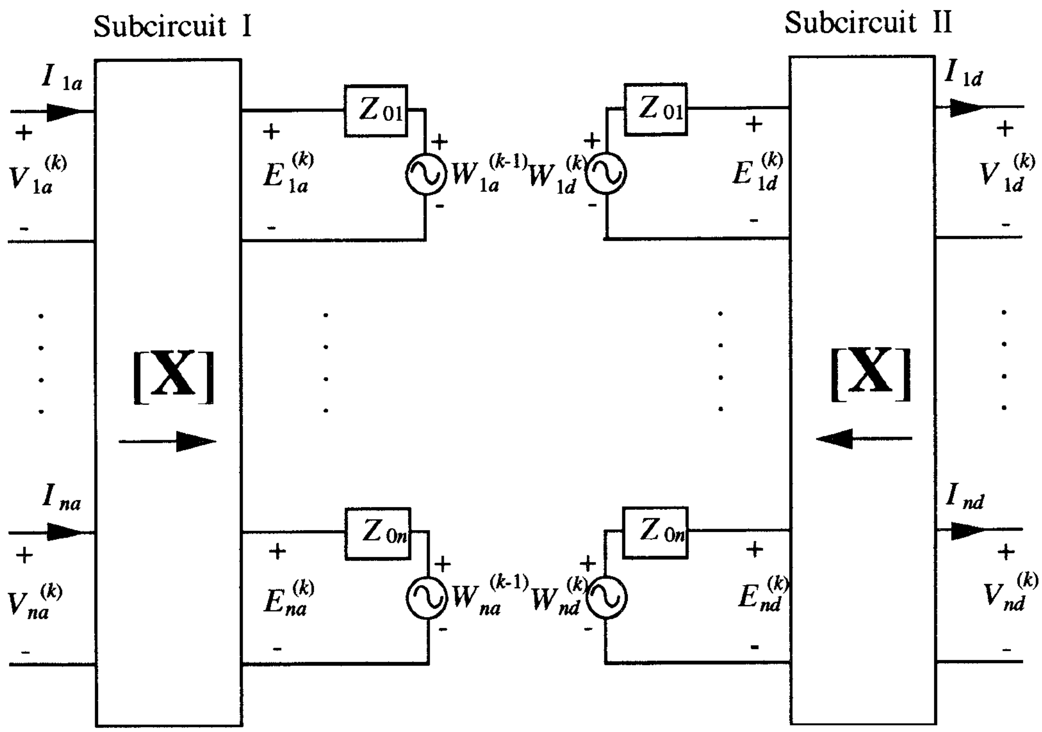

Fig. 2. Equivalent model of the lossy coupled transmission-line system in Fig. 1.

Step 6a: Evaluate $e_a^{(k)}(t)$ using

$$e_a^{(k)}(t) = X^{-1} v_a^{(k)}(t)$$

and compute the voltage generators $w_{id}^{(k-1/2)}(t)$, $i = 1, 2, \cdots, n$ by the FFT and the inverse FFT:

$$w_{id}^{(k-1/2)}(t) = \text{IFFT}\{\exp(-\theta_i) * \text{FFT}[2e_{ia}^{(k)}(t) - w_{ia}^{(k-1)}(t)]\}.$$

Step 6b: Shift the waveforms of the voltage generators $w_{id}^{(k-1/2)}(t)$ by the values $\{w_{id0} - w_{id}^{(k-1/2)}(0)\}$, $i = 1, 2, \cdots, n$ such that the initial dc conditions are satisfied, i.e.

$$w_{id}^{(k)}(t) = w_{id}^{(k-1/2)}(t) + \{w_{id0} - w_{id}^{(k-1/2)}(0)\}$$

where $w_{id0]}$ is the ith element of the vector w_{d0}.

Step 7: Connect the Thevenin equivalent circuit $\{E_Y, Z_Y\}$ to Subcircuit II and carry out the transient analysis for the entire time interval ($0 \leq t \leq T$) to obtain the terminal voltage waveforms $\{v_d^{(k)}(t)\}$.

Step 8a: Evaluate $e_d^{(k)}(t)$ using

$$e_d^{(k)}(t) = X^{(-1)} v_d^{(k)}(t)$$

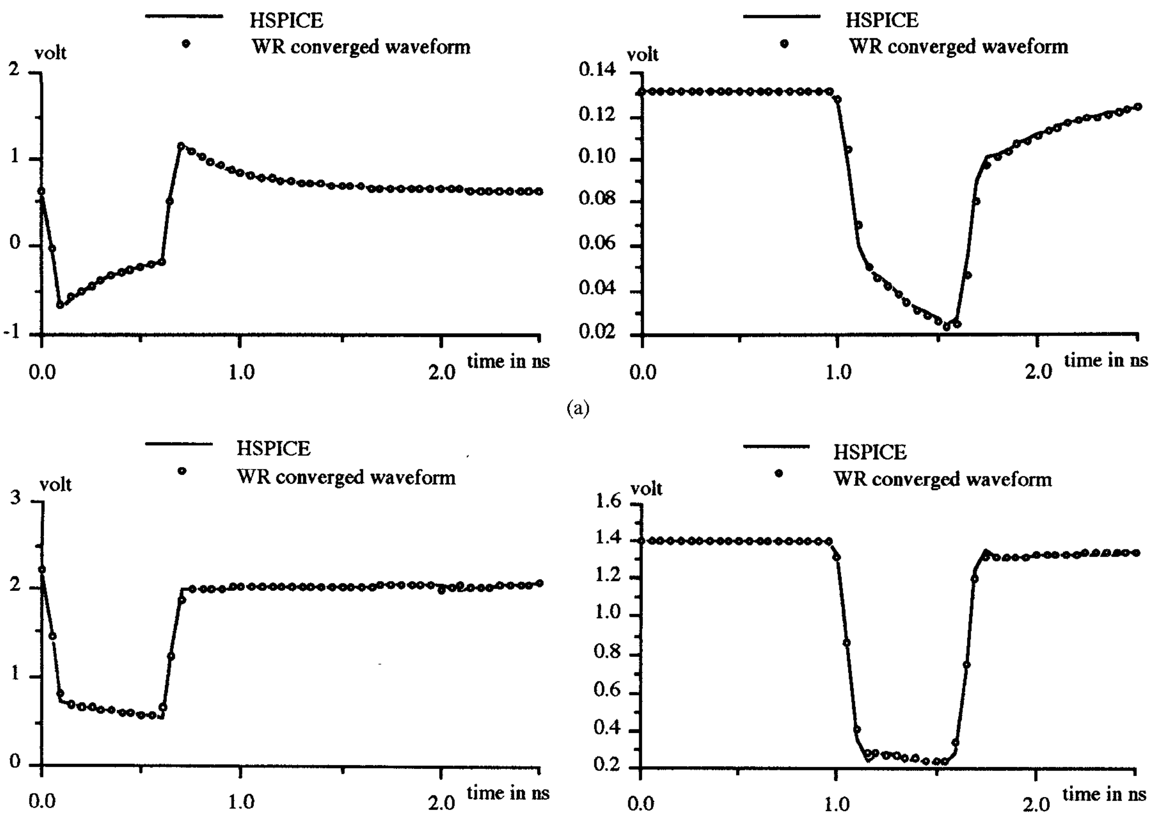

Fig. 3. (a) Iterative waveform simulation of near-end terminal voltage waveforms (left) and far-end terminal voltage waveforms (right) of the active outer transmission line using waveform relaxation algorithm with $\boldsymbol{G}$ = diag ($1e-2$, $1e-2$, $1e-2$) /Ω/cm. (b) Iterative waveform simulation of near-end terminal voltage waveforms (left) and far-end terminal voltage waveforms (right) of the active outer transmission line using waveform relaxation algorithm with negligible dielectric loss.

and evaluate the voltage generators $w_{ia}^{(k-1/2)}(t)$, $i = 1, 2, \cdots, n$ by the FFT and the inverse FFT

$$w_{ia}^{(k-1/2)}(t) = \mathrm{IFFT}\{\exp(-\theta_i) * \mathrm{FFT}[2e_{id}^{(k)}(t) - w_{id}^{(k)}(t)]\}.$$

Step 8b: Shift the waveforms of the voltage generators $w_{ia}^{(k-1/2)}(t)$ by the values $\{w_{ia0} - w_{ia}^{(k-1/2)}(0)\}$, $i = 1, 2, \cdots, n$ such that the initial dc conditions are satisfied, i.e.

$$w_{ia}^{(k)}(t) = w_{ia}^{(k-1/2)}(t) + \{w_{ia0} - w_{ia}^{(k-1/2)}(0)\}$$

.

Step 9: Stop the iteration if the iteration count exceeds a preset integer number or if the difference between the results obtained in successive iterations is sufficiently small. Otherwise, set $k = k + 1$ and go to Step 5 to repeat the iteration process.

Although it is not strictly necessary to shift the waveforms of the generators evaluated by the Fourier transforms as described in Steps 6b and 8b, this is done so that the initial conditions are satisfied and to enable the algorithm to converge faster.

Convergence Theorem of Waveform Relaxation Algorithm: For an n-conductor lossy stripline system terminating in the Thevenin equivalent circuits and satisfying the condition $\boldsymbol{RG} = m\boldsymbol{I}_n$, the Waveform Relaxation Algorithm generates a sequence of waveforms $\{\boldsymbol{v}_a^{(k)}(t), \boldsymbol{v}_d^{(k)}(t)\}$ converging to the exact solution of the terminal voltages given by (6) and (7). This may be proved by a method similar to that given in [1].

III. Some Simulation Results

A triconductor stripline system has been analysed with the parameters given in Appendix A. The dielectric loss has been chosen deliberately to be large enough to be effective. The system has also been simulated by the conventional circuit simulator HSPICE where the coupled lossy transmission lines are represented by 100 symmetrical blocks (Appendix B) connected in tandem. Extremely small time steps of 0.2 ps have been used in HSPICE to eliminate false ringing effects.

The iterative waveforms at the ends of the active outer transmission line are compared, as in Fig. 3(a). The waveforms generated by the waveform relaxation algorithm (**WR**) in the first iteration actually give the same results with those given by HSPICE. The second iteration, which gives identical waveforms, is performed only to ensure convergence . HSPICE has taken about 5150 seconds to simulate the above system while the waveform relaxation algorithm takes only 6.1 s to perform the two iterations, more than 800 times faster. This again demonstrates the extreme efficiency and accuracy that waveform relaxation can provide. Fig. 3(b) shows the simulated waveforms when the dielectric loss is negligible.

IV. Conclusion

An iterative method based on waveform relaxation has been proposed to simulate a set of coupled lossy transmission lines embedded in a lossy medium. When the dielectric loss to ground is much more significant than the loss between transmission lines, and the resistive and dielectric loss matrices satisfy the relation $\boldsymbol{RG} = m\boldsymbol{I}_n$, the method decouples the lines by using two congruence transformers at the ends of the lines. It is accurate and effective regardless of the magnitude of the dielectric loss. The speed improvement has been demonstrated to be several hundred times compared with the conventional method HSPICE. The algorithm is particularly suited to

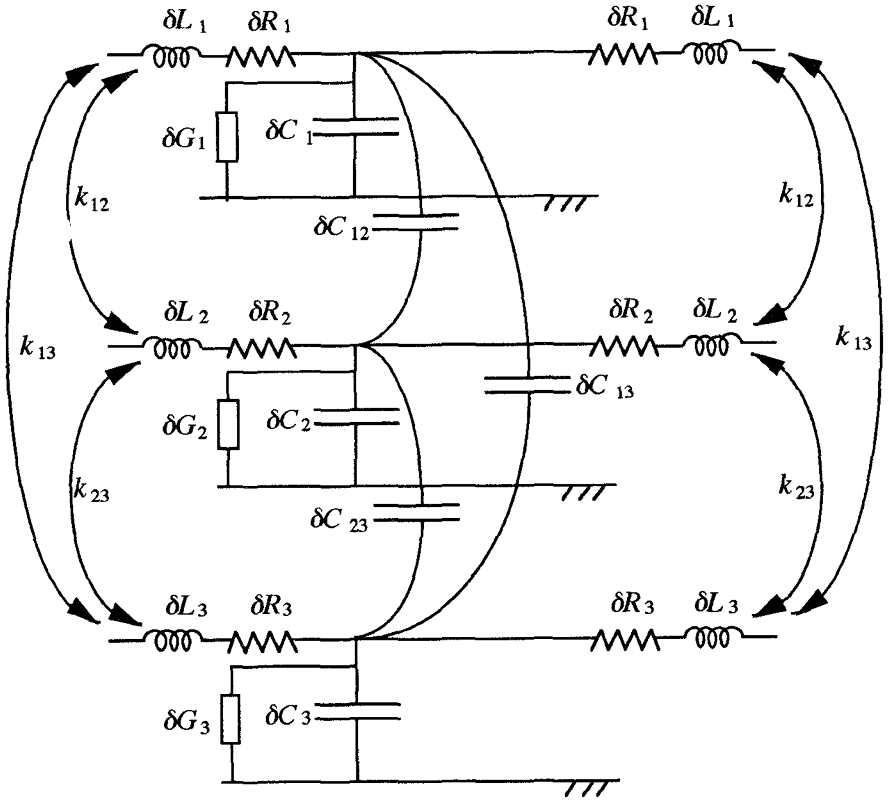

Fig. 4. Symmetrical block used in modelling coupled lossy transmission lines in HSPICE.

identical lines with negligible di electric loss between them and are the same distance from the ground plane. It can also be applied to coupled lines with the resistive and dielectric loss matrices satisfying the relation $\boldsymbol{RG} = m\boldsymbol{I}_n$.

Appendix A: Triconductor Stripline Parameters

The triconductor stripline system has been analysed with the following parameters

$$\boldsymbol{L} = \begin{bmatrix} 3 & 1 & 0.5 \\ 1 & 4 & 1 \\ 0.5 & 1 & 3 \end{bmatrix} nH/\text{cm}$$

$$\boldsymbol{C} = \left(\frac{1}{24}\right) \begin{bmatrix} 44 & -10 & -4 \\ -10 & 35 & -10 \\ -4 & -10 & 44 \end{bmatrix} \text{pF/cm}$$

$$\tilde{\boldsymbol{L}} = \text{diag}(5/3, 5, 2.5) nH/\text{cm}$$

$$\boldsymbol{R} = \text{diag}(2.0, 2.0, 2.0)\Omega/\text{cm}$$

$$\boldsymbol{G} = \text{diag}(1e-2, 1e-2, 1e-2)/\Omega/\text{cm}$$

$$\text{length} l = 14.14\,\text{cm}$$

$$\boldsymbol{Z}_X = \text{diag}(100, 200, 100)\Omega$$

$$\boldsymbol{Z}_Y = \text{diag}(50, 75, 50)\Omega$$

For the Thevenin equivalent voltage sources,

$$e_{2X}(t) = e_{3X}(t) = e_{1Y}(t) = e_{2Y}(t) = e_{3Y}(t) = 0.0 \text{ for all } t$$

and $e_{1X}(t)$ is an inverted pulse with value 5V at $t = 0$, falling linearly to 0V at $t = 100$ ps, staying at 0V until $t = 600$ ps, and starting rising linearly to 5V at $t = 700$ ps, then remaining at 5V for the rest of the time.

Appendix B: Symetrical *RCL* Block Used in HSPICE

Fig. 4 shows the symmetrical block used in modelling coupled lossy transmission lines in HSPICE. The values of the lumped elements in the block are given by

$$\delta L_i = \frac{L_{ii} * l}{2 * \text{no of symmetrical blocks}} \qquad i = 1, 2, 3$$

$$L_{ii} = i\text{th diagonal element of } \boldsymbol{L}$$

$$\delta C_i = \frac{\left(\sum_{j=1}^{3} C_{ij}\right) * l}{\text{no. of symmetrical blocks}} \qquad i = 1, 2, 3$$

$$C_{ij} = i, j\text{th element of } \boldsymbol{C}$$

$$\delta C_{ij} = \frac{|C_{ij}| * l}{\text{no. of symmetrical blocks}} \qquad 1 \le i < j \le 3$$

$$C_{ij} = i, j\text{th element of } \boldsymbol{C}$$

$$\delta R_i = \frac{R_{ii} * l}{2 * \text{no. of symmetrical blocks}} \qquad i = 1, 2, 3$$

$$R_{ii} = i\text{th diagonal element of } \boldsymbol{R}$$

$$\delta G_i = \frac{G_{ii} * l}{\text{no. of symmetrical blocks}} \qquad i = 1, 2, 3$$

$$G_{ii} = i\text{th diagonal element of } \boldsymbol{G}$$

and the mutual inductance coefficients are

$$k_{ij} = \frac{L_{ij}}{\sqrt{L_i L_j}} \qquad 1 \le i < j \le 3$$

$$L_{ij} = i, j\text{th element of } \boldsymbol{L}.$$

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A Useful Theorem for a Lossless Multiport Network

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***Abstract*—A useful theorem is obtained for a lossless multiport network from the unitary condition of scattering matrix, and is proven to be equivalent to the unitary condition. Some illustrations are given to show how to apply the theorem to the analysis of the properties of a lossless *n*-port network.**

I. INTRODUCTION

In microwave engineering, many passive components can be taken as lossless. Therefore, analysis and synthesis of lossless networks are very important.

It is well known that the scattering matrix S of a lossless network meets the unitary condition

$$S^{+}S = I. \tag{1}$$

For a lossless two-port network, the following constrained conditions may result from (1) [1]:

$$|S_{11}| = |S_{22}|, \qquad |S_{12}| = |S_{21}| \tag{2}$$

$$\exp[j(\varphi_{11} + \varphi_{22})] = \exp\{j[(\varphi_{12} + \varphi_{21}) + \pi]\} = \det(S) \tag{3}$$

$$|S_{11}|^2 + |S_{21}|^2 = |S_{22}|^2 + |S_{12}|^2 = 1 \tag{4}$$

in which $S_{ik}(i, k = 1, 2)$ is the element of matrix $S, j = \sqrt{-1}, \varphi_{ik}$ is the phase of the element S_{ik}, $\det(S)$ means the determinant of matrix S.

For a lossless n-port network $(n > 2)$, things become complex. Multiplying S^{+} with S and demanding the product-matrix equal unitary matrix, we can get n real equations and $n(n-1)/2$ complex equations. Those equations appear in a form different from (2) and (3). Also, they are not convenient to the analysis of the properties of a lossless n-port network.

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In 1991, Liang and Qiu [2] first found that the magnitude relation (2) may be generalized to a lossless n-port network. This paper shows that the phase relation (3) may also be generalized to a lossless n-port network. Furthermore, while the matrix S of a lossless n-port network $(n > 2)$ meets the generalized magnitude relation and generalized phase relation, any column (or row) of matrix S must be a complex unit vector (by using the term complex unit vector, we mean that the square sum of the magnitude of its elements equals 1), which is the generalized form of (4). That is to say, for a lossless n-port network $(n > 2)$, the generalized magnitude relation and phase relation are equivalent to the unitary condition (1). To be more important and meaningful, it is found that by using the generalized magnitude relation and generalized phase relation, the analysis of the properties of a lossless n-port network becomes much simpler. Three illustrations are given in this paper.

II. TWO THEOREMS FOR LOSSLESS NETWORKS

Because the magnitude of the determinant of a scattering matrix must be 1 for lossless networks, in this paper, we will always let

$$\det(S) = \exp(j\varphi_D) \tag{5}$$

for a lossless network, where φ_D is the phase of the determinant of scattering matrix S.

Theorem 1: For a lossless n-port network, write its scattering matrix S in partitioned form

$$S = \begin{bmatrix} S_{aa} & S_{ab} \\ S_{ba} & S_{bb} \end{bmatrix}_{n\times n} \tag{6}$$

so that at least one of the two submatrix pairs (S_{aa}, S_{bb}), (S_{ab}, S_{ba}) is a square matrix pair. Let M_{ik}^{c} represent the cofactor of square submatrix $S_{ik}(i, k = a, b)$ in $\det(S)$; then we have

$$|\det(S_{ik})| = |M_{ik}^{c}| = |\det(S_{ki})| \qquad (i, k = a, b) \tag{7}$$

$$\arg[\det(S_{ik})] + \arg(M_{ik}^{c}) = \varphi_D \qquad (i, k = a, b) \tag{8}$$

or, equivalently,

$$[\det(S_{ik})]^{*} = \exp(-j\varphi_D)M_{ik}^{c} \qquad (i, k = a, b). \tag{9}$$

Remark: Applying Theorem 1 to a two-port network, we can get (2) from (7) and (3) from (8). Therefore, we call (7) and (8) the generalized magnitude relation and generalized phase relations, respectively. For convenience, in the following we will make use of (9) instead of (7) and (8).

Proof: The proof will be given only for the case that (S_{ab}, S_{ba}) is a square submatrix pair. A similar proof may be easily made for other cases.

Suppose S_{ab} is an $m \times m$ matrix. Then, S_{ba} is an $(n-m)\times(n-m)$ matrix, S_{aa} is an $m \times (n-m)$ matrix, S_{bb} is an $(n-m) \times m$ matrix. By applying the unitary condition (1) to partitioned matrix (6), we can get

$$\begin{aligned} S_{aa}^{+}S_{aa} + S_{ba}^{+}S_{ba} &= I_{n-m} \\ S_{ab}^{+}S_{ab} + S_{bb}^{+}S_{bb} &= I_{m} \\ S_{aa}^{+}S_{ab} + S_{ba}^{+}S_{bb} &= O_{(n-m)\times m} \end{aligned} \tag{10}$$

where I_m represents the $m \times m$ unit matrix, $O_{(n-m)\times m}$ represents an $(n-m) \times m$ zero matrix.